# A REFINEMENT OF REACCELERATED OVER RELAXATION (RROR) METHOD

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**ABSTRACT**: In this paper, we discussed a Refinement of Reaccelerated over Relaxation (RROR) method for solving linear system of equations is introduced and also, it is shown that this method is superior to the well known ROR, SOR, Gauss-Seidal and Jacobi methods through some numerical examples.

**KEY WORDS:**Jacobi method, Refinement of Jacobi method, Gauss-Seidal method, Refinement of Gauss-Seidal method, Refinement of SOR method, ROR method, Diagonally Dominant Matrix, Spectral radius

## 1. INTRODUCTION:

For solving numerically the linear system of equations

$$AX = b \ldots (1.1)$$

where A is non-singular with non-vanishing diagonal elements of over $n \times n$, $X$ and $b$ are unknown and known n-dimensional vectors. We split the coefficient matrix 'A'without any loss of generality, as

$$A = D - L - U \ldots (1.2)$$

Where $D$ is the diagonal matrix, L and U are strictly lower and upper triangular parts of A,
Then the system (1.1) takes the form

$$AX = \hat{b} \qquad \ldots (1.3)$$

The Accelerated over relaxation(AOR) method for solving (1.3) is

$$X^{(n+1)} = \left(I - \omega L\right)^{-1}\left\{\left(1-r\right)I + \left(r-\omega\right)L + rU\right\}X^{(n)} + r\left(I - \omega L\right)^{-1}\hat{b} \ldots (1.4)$$

The method's Successive over relaxation (SOR), Gauss-Seidal(G.S) and Jacobi can be realised from (1.4) for the choice of $r$ and $\omega$ as

$$\left(r,\omega\right),\left(\omega,\omega\right),\left(1,1\right),\left(1,0\right) \ldots (1.5)$$

The iteration matrices of the above matrices are:

$$\text{AOR}= M_{r,\omega} = \left(I - \omega L\right)^{-1}\left\{\left(1-r\right)I + \left(r-\omega\right)L + rU\right\} \ldots (1.6)$$

$$\text{SOR}= S_{\omega} = \left(I - \omega L\right)^{-1}\left\{\left(1-\omega\right)I + \omega U\right\} \quad \ldots (1.7)$$

$$\text{Gauss-Seidal=G.S} = \left(I - L\right)^{-1}U \ldots (1.8)$$

$$\text{Jacobi} = J = \left(L + U\right) \ldots (1.9)$$

whose spectral radii are given by

$$S\left(M_{r,\omega}\right)=\begin{cases}\dfrac{\overline{\mu}^2}{\left(1+\sqrt{1-\overline{\mu}^2}\right)^2} & \text{when } \underline{\mu}=0 \ (\text{or}) \ \sqrt{1-\overline{\mu}^2}\le 1-\underline{\mu}^2 \ (\text{or})\ 0<\underline{\mu}<\overline{\mu} \ \ldots(1.9.1)\\ 0 & \text{when } \underline{\mu}=\overline{\mu} \ \ldots(1.9.2)\\ \dfrac{\underline{\mu}\sqrt{\overline{\mu}^2-\underline{\mu}^2}}{\sqrt{1-\underline{\mu}^2}\left(1+\sqrt{1-\overline{\mu}^2}\right)} & \text{when } 0<\underline{\mu}<\overline{\mu} \ (\text{or})\, 1-\underline{\mu}^2<\sqrt{1-\overline{\mu}^2} \ \ldots(1.9.3)\end{cases}$$

$$S\left(S_\omega\right)=\frac{\overline{\mu}^2}{\left(1+\sqrt{1-\overline{\mu}^2}\right)^2} \ldots (1.10)$$

$$S\left(G.S\right)=\overline{\mu}^2 \ldots (1.11)$$

$$S\left(J\right)=\overline{\mu} \ldots (1.12)$$

Where $\underline{\mu}$ and $\overline{\mu}$ are the smallest and the largest eigen values of Jacobi matrix in magnitude.

Reaccelerated over relaxation (ROR) method for solving (1.3) developed by V.B.Kumar, Vatti et.al [5] is given by

$$X^{(n+1)}=R_{r,w}X^{(n)}+\left(r-r\omega\right)\left(I-\omega L\right)^{-1}\hat{b} \ldots (1.13)$$

Whose iteration matrix is

$$R_{r,\omega}=\left(I-\omega L\right)^{-1}\left\{\left(1-r+r\omega\right)I+\left(r-\omega-r\omega\right)L+\left(r-r\omega\right)U\right\} \ldots (1.14)$$

## 2. REFINEMENT OF REACCELERATED OVER RELAXATION(RROR) METHOD

Multiply both sides of the equation (1.3) by $\left(r-r\omega\right)$ we obtain

$$\left(r-r\omega\right)AX=\left(r-r\omega\right)\hat{b}$$

$$\left(I-\omega L\right)X=\left(I-\omega L\right)X+\left(r-r\omega\right)\left(\hat{b}-AX\right)$$

$$X=X+\left(r-r\omega\right)\left(I-\omega L\right)^{-1}\left(\hat{b}-AX\right) \ldots (2.1)$$

Now, the refinement of ROR method is defined as

$$X^{(n+1)}=X^{(n+1)}+\left(r-r\omega\right)\left(I-\omega L\right)^{-1}\left(\hat{b}-AX^{(n+1)}\right) \ldots (2.2)$$

$$X^{(n+1)} = R_{r,w}X^{(n)} + (r - r\omega)(I - \omega L)^{-1}\hat{b}$$

$$+(r - r\omega)(I - \omega L)^{-1}\left(\hat{b} - (I - L - U)\left(R_{r,w}X^{(n)} + (r - r\omega)(I - \omega L)^{-1}\hat{b}\right)\right)$$

$$\Rightarrow X^{(n+1)} = R_{r,\omega}X^{(n)} + 2(r - r\omega)(I - \omega L)^{-1}\hat{b}$$

$$-(I - \omega L)^{-1}\left((r - r\omega)I - (r - r\omega)L - (r - r\omega)U\right)\left(R_{r,\omega}X^{(n)} + (r - r\omega)(I - \omega L)^{-1}\hat{b}\right)$$

$$\Rightarrow X^{(n+1)} = R_{r,\omega}X^{(n)} + 2(r - r\omega)(I - \omega L)^{-1}\hat{b}$$

$$-(I - \omega L)^{-1}\left((I - I + \omega L - \omega L) + (r - r\omega)I - (r - r\omega)L - (r - r\omega)U\right)\left(R_{r,\omega}X^{(n)} + (r - r\omega)(I - \omega L)^{-1}\hat{b}\right)$$

$$\Rightarrow X^{(n+1)} = R_{r,\omega}X^{(n)} + 2(r - r\omega)(I - \omega L)^{-1}\hat{b}$$

$$-(I - \omega L)^{-1}\left((I - \omega L) - \left\{(1 - r + r\omega)I + (r - \omega - r\omega)L + (r - r\omega)U\right\}\right)\left(R_{r,\omega}X^{(n)} + (r - r\omega)(I - \omega L)^{-1}\hat{b}\right)$$

$$\Rightarrow X^{(n+1)} = R^2_{r,\omega}X^{(n)} + (r - r\omega)R_{r,\omega}(I - \omega L)^{-1}\hat{b} + (r - r\omega)(I - \omega L)^{-1}\hat{b}$$

$$\Rightarrow X^{(n+1)} = R^2_{r,\omega}X^{(n)} + (r - r\omega)(I - \omega L)^{-1}\hat{b}(I + R_{r,\omega}) \quad \ldots (2.3)$$

$$(n = 0, 1, 2, \ldots)$$

$$\Rightarrow X^{(n+1)} = \overline{P}X^{(n)} + \overline{Q}$$

is the refinement of ROR method for the solution of (1.3)

Where $\overline{P} = P^2, \overline{Q} = (r - r\omega)(I + P)Q$

Here $P = R_{r,\omega}$ and $Q = (I - \omega L)^{-1}\hat{b}$

## 3. CONVERGENCE OF REFINEMENT OF ACCELERATED OVER RELAXATION(RROR) METHOD

Theorem 3.1: Let A be irreducible matrix with weak diagonal dominance. Then RROR method converges for any arbitrary choice of the initial approximation.

Proof:Let $X^*$ be the exact solution and if $\overline{X}^{(n+1)}$ be the (n+1)$^{th}$approximation to the solution of (1.3) by the method (2.2)

Now

$$\left\|\overline{X}^{(n+1)} - X^*\right\| = \left\|X^{(n+1)} + (r - r\omega)(I - \omega L)^{-1}(\hat{b} - AX) - X^*\right\|$$

$$\leq \left\|X^{(n+1)} - X^*\right\| + \left\|(\hat{b} - AX)\right\|\left\|(r - r\omega)(I - \omega L)^{-1}\right\|$$

$$\left(\because \left\|X^{(n+1)} - X^*\right\| \to 0 \text{ and } \left\|(\hat{b} - AX)\right\| \to 0\right)$$

$\therefore \left\| \overline{X}^{(n+1)} - X^* \right\| \to 0$

Therefore, refinement of ROR method converges to the solution of the linear system (1.3)

Theorem 3.2: if A is irreducible matrix with weak diagonal dominance , then $\left\| \overline{P} \right\|_\infty = \left\| P \right\|^2_\infty < 1$

Proof: $\left\| \overline{P} \right\|_\infty = \left\| R_{r,\omega}{}^2 \right\|_\infty = \left\| R_{r,\omega}{}^2 \right\|^2_\infty = \left\| P \right\|^2_\infty < 1$

Theorem 3.3: if A is irreducible matrix with weak diagonal dominance, then $\left\| \overline{P} \right\|_\infty = \left\| P \right\|_\infty$

Proof: $\left\| \overline{P} \right\|_\infty = \left\| R_{r,\omega}{}^2 \right\|_\infty$

$= \left\| R_{r,\omega}{}^2 \right\|^2_\infty$

$= \left\| P \right\|^2_\infty < \left\| P \right\|_\infty$

4. **NUMERICAL EXAMPLES**:

Example 4.1: If $A = \begin{bmatrix} 1 & 0 & \frac{1}{5} & \frac{1}{5} \\ 0 & 1 & -\frac{71}{10} & \frac{113}{10} \\ \frac{16}{5} & \frac{1}{5} & 1 & 0 \\ 2 & \frac{1}{5} & 0 & 1 \end{bmatrix}$ which is considered by G.Avdelas and A. Hadjimos [1]

and $b = \begin{bmatrix} 1.4 \\ 5.2 \\ 4.4 \\ 3.2 \end{bmatrix}$ then the eigenvalues of the Jacobi matrix are $\pm\frac{\sqrt{23}}{5}$ and $\pm\frac{\sqrt{24}}{5}$ and hence $\underline{\mu} = \frac{\sqrt{23}}{5}$, $\overline{\mu} = \frac{\sqrt{24}}{5}$ .It can be seen that the condition given by G. Avdelas and A.Hadjidimos [1] i.e., $0 < \underline{\mu} < \overline{\mu}$ and $1 - \underline{\mu}^2 < \sqrt{1 - \overline{\mu}^2}$ are satisfied.Spectral radius ofAOR iteration matrix is $\frac{\sqrt{246}}{12}$ but not $\frac{\sqrt{46}}{12}$

**Spectral radius of different methods with their choices**

| S.No. | Method | Choices of parameters | Spectral radius |
|---|---|---|---|
| 1 | ROR | $\omega = \frac{5}{3}, r = -4.03$ | 0.56892569 |
| 2 | RROR | $\omega = \frac{5}{3}, r = -4.03$ | 0.32367644 |
| 3 | AOR | $\omega = \frac{5}{3}, r = \frac{14}{3}$ | 0.75129518 |

| | | | |
|---|---|---|---|
| 4 | RAOR | $\omega=\frac{5}{3}, r=\frac{14}{3}$ | 0.56444445 |
| 5 | SOR | $\omega=\frac{5}{3}, r=\frac{5}{3}$ | $\frac{2}{3}$ |
| 6 | RSOR | $\omega=\frac{5}{3}, r=\frac{5}{3}$ | $\frac{4}{9}$ |
| 7 | G.S | - | 0.96 |
| | RG.S | - | 0.9216 |
| 8 | J | - | 0.97979590 |
| 9 | RJ | - | 0.96000001 |

## 5. CONCLUSION:

Reducing the spectral radius of the iteration matrix corresponds to an increased rate of convergence for the numerical solution of the system of linear equations.

$$S\left(R_{r,\omega}{}^{2}\right) < S\left(R_{r,\omega}\right) < S\left(M_{r,\omega}{}^{2}\right) < S\left(M_{r,\omega}\right) < S\left(S_{\omega}{}^{2}\right) < \rho\left(S_{\omega}\right)$$

$$< S\left(G.S^{2}\right) < S\left(G.S\right) < S\left(J^{2}\right) < S\left(J\right) < 1$$

## REFERENCES:


1. G. Avdelas and A. Hadjimos, Optimum accelerated over relaxation method in a special case. Math.Comp., 36(153)(1981), 183-187.
2. V. B. Kumar Vatti, Ramadevi, M. S. Kumar Mylapalli, (2018) A Refinement of Accelerated Over Relaxation Method for the Solution of Linear System. International Journal of Pure and Applied mathematics. Vol. 118, No. 18, pp. 1571-1577.
3. Apostolos Hadjidimos, (January 1978). Accelerated over relaxation method. Mathematics of computations, Vol. 32, No.141, pp. 149-157.
4. V. B. Kumar Vatti and Tesfaye Kebede Eneyew, (2011). A refinement of Gauss-Seidel method for solving a linear system of equations, Int. J. Contemp. Math. Sciences, Vol. 6, No. 3,pp. 117-121.
5. V. B. Kumar Vatti, G.Chinna Rao, Reaccelerated Over Relaxation (ROR) Method. Bull.Int.Math. Virtual Inst., Vol.10 (2) (2020).315-324.
6. David M. Young, Iterative solutions of Large Linear Systems, New York and London, 1971.